标架变换（续）：随流导数

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△标架变换下的时间导数：若向量场 $\vec{a}(\vec{r},t)$ 具有标架变换不变性，即满足 $\vec{a}^*(\vec{r}^*,t^*)=\underline{\underline{Q}}(t)\vec{a}^*(\vec{r},t)$，则其时间偏导数一般不满足标架不变性，因为

$$\frac{\partial}{\partial t}\vec{a}^*(\vec{r}^*,t^*)=\frac{\partial}{\partial t}\left[\underline{\underline{Q}}(t)\vec{a}(\vec{r},t)\right]=\dot{\underline{\underline{Q}}}\vec{a}+\underline{\underline{Q}}\dot{\vec{a}}\neq\underline{\underline{Q}}\dot{\vec{a}}$$

△为定义一种保持标架变换不变性的时间导数，我们先推导一些关系式.

由 $\underline{\underline{L}}^*=\underline{\underline{Q}}\,\underline{\underline{L}}\,\underline{\underline{Q}}^T+\dot{\underline{\underline{Q}}}\underline{\underline{Q}}^T$，$(\underline{\underline{L}}^*-\underline{\underline{Q}}\,\underline{\underline{L}}\,\underline{\underline{Q}}^T)\underline{\underline{Q}}=\boxed{\dot{\underline{\underline{Q}}}=\underline{\underline{L}}^*\underline{\underline{Q}}-\underline{\underline{Q}}\,\underline{\underline{L}}}$ $\boxed{\dot{\underline{\underline{Q}}}^T=\underline{\underline{Q}}^T\underline{\underline{L}}^{*T}-\underline{\underline{L}}^T\underline{\underline{Q}}^T}$

又由 $\underline{\underline{Q}}\,\underline{\underline{Q}}^T=\underline{\underline{I}}$，

$$\frac{\partial}{\partial t}(\underline{\underline{Q}}\,\underline{\underline{Q}}^T)=\underline{0}=\dot{\underline{\underline{Q}}}\underline{\underline{Q}}^T+\underline{\underline{Q}}\dot{\underline{\underline{Q}}}^T\Leftrightarrow\dot{\underline{\underline{Q}}}\underline{\underline{Q}}^T=-\underline{\underline{Q}}\dot{\underline{\underline{Q}}}^T$$

∴ $\underline{\underline{L}}^*=\underline{\underline{Q}}\,\underline{\underline{L}}\,\underline{\underline{Q}}^T-\underline{\underline{Q}}\dot{\underline{\underline{Q}}}^T$，$\underline{\underline{Q}}^T(\underline{\underline{Q}}\,\underline{\underline{L}}\,\underline{\underline{Q}}^T-\underline{\underline{L}}^*)=\boxed{\dot{\underline{\underline{Q}}}^T=\underline{\underline{L}}\,\underline{\underline{Q}}^T-\underline{\underline{Q}}^T\underline{\underline{L}}^*}$ $\boxed{\dot{\underline{\underline{Q}}}=\underline{\underline{Q}}\,\underline{\underline{L}}^T-\underline{\underline{L}}^{*T}\underline{\underline{Q}}}$

△由上面的关系式，

$$\frac{\partial}{\partial t}\vec{a}^*(\vec{r}^*,t^*)=\dot{\underline{\underline{Q}}}\vec{a}+\underline{\underline{Q}}\dot{\vec{a}}=(\underline{\underline{Q}}\,\underline{\underline{L}}^T-\underline{\underline{L}}^{*T}\underline{\underline{Q}})\vec{a}+\underline{\underline{Q}}\dot{\vec{a}}$$

$$=\underline{\underline{Q}}\,\underline{\underline{L}}^T\vec{a}-\underline{\underline{L}}^{*T}\vec{a}^*+\underline{\underline{Q}}\dot{\vec{a}}$$

$\Leftrightarrow$ $\dot{\vec{a}}^*+\underline{\underline{L}}^{*T}\vec{a}^*=\underline{\underline{Q}}(\dot{\vec{a}}+\underline{\underline{L}}^T\vec{a})$，可见，客观向量场的表达式 $\dot{\vec{a}}+\underline{\underline{L}}^T\vec{a}$ 总是客观的。故定义：

$\overset{\circ}{\vec{a}}(\vec{r},t)\overset{def}{=}\frac{\partial}{\partial t}\vec{a}(\vec{r},t)+\underline{\underline{L}}^T(\vec{r},t)\vec{a}(\vec{r},t)$ 或简写成 $\overset{\triangle}{\vec{a}}\overset{def}{=}\dot{\vec{a}}+\underline{\underline{L}}^T\vec{a}$ 上随流

则共旋导数保持张量场的标架不变性，采用 $\dot{\underline{\underline{Q}}}$ 的另一表达式，还可定义：$\overset{\triangledown}{\vec{a}}\overset{def}{=}\dot{\vec{a}}-\underline{\underline{L}}\vec{a}$ 下随流.

△若张量场 $\underline{\underline{T}}(\vec{r},t)$ 具有标架变换不变性，即满足：

$$\underline{\underline{T}}^*(\vec{r}^*,t^*)=\underline{\underline{Q}}(t)\,\underline{\underline{T}}(\vec{r},t)\,\underline{\underline{Q}}^T(t)$$

则其时间偏导数 $\frac{\partial}{\partial t}\underline{\underline{T}}^*=\dot{\underline{\underline{Q}}}\,\underline{\underline{T}}\,\underline{\underline{Q}}^T+\underline{\underline{Q}}\,\dot{\underline{\underline{T}}}\,\underline{\underline{Q}}^T+\underline{\underline{Q}}\,\underline{\underline{T}}\,\dot{\underline{\underline{Q}}}^T\neq\underline{\underline{Q}}\,\dot{\underline{\underline{T}}}\,\underline{\underline{Q}}^T$

$$=(\underline{\underline{L}}^*\underline{\underline{Q}}-\underline{\underline{Q}}\,\underline{\underline{L}})\underline{\underline{T}}\,\underline{\underline{Q}}^T+\underline{\underline{Q}}\,\dot{\underline{\underline{T}}}\,\underline{\underline{Q}}^T+\underline{\underline{Q}}\,\underline{\underline{T}}(\underline{\underline{Q}}^T\underline{\underline{L}}^{*T}-\underline{\underline{L}}^T\underline{\underline{Q}}^T)$$

$$=\underline{\underline{L}}^*\underline{\underline{Q}}\,\underline{\underline{T}}\,\underline{\underline{Q}}^T-\underline{\underline{Q}}\,\underline{\underline{L}}\,\underline{\underline{T}}\,\underline{\underline{Q}}^T+\underline{\underline{Q}}\,\dot{\underline{\underline{T}}}\,\underline{\underline{Q}}^T+\underline{\underline{Q}}\,\underline{\underline{T}}\,\underline{\underline{Q}}^T\underline{\underline{L}}^*-\underline{\underline{Q}}\,\underline{\underline{T}}\,\underline{\underline{L}}^T\underline{\underline{Q}}^T$$

$$=\underline{\underline{L}}^*\underline{\underline{T}}^*-\underline{\underline{Q}}\,\underline{\underline{L}}\,\underline{\underline{T}}\,\underline{\underline{Q}}^T+\underline{\underline{Q}}\,\dot{\underline{\underline{T}}}\,\underline{\underline{Q}}^T+\underline{\underline{T}}^*\underline{\underline{L}}^*-\underline{\underline{Q}}\,\underline{\underline{T}}\,\underline{\underline{L}}^T\underline{\underline{Q}}^T$$

$$\Leftrightarrow\quad\dot{\underline{\underline{T}}}^*-\underline{\underline{L}}^*\underline{\underline{T}}^*-\underline{\underline{T}}^*\underline{\underline{L}}^*=\underline{\underline{Q}}(\dot{\underline{\underline{T}}}-\underline{\underline{L}}\,\underline{\underline{T}}-\underline{\underline{T}}\,\underline{\underline{L}})\underline{\underline{Q}}^T$$

可见客观张量场的表达式 $\dot{\underline{\underline{T}}}-\underline{\underline{L}}\,\underline{\underline{T}}-\underline{\underline{T}}\,\underline{\underline{L}}$ 总是客观的，故定义：

$$\overset{\triangledown}{\underline{\underline{T}}}=\dot{\underline{\underline{T}}}-\underline{\underline{L}}\,\underline{\underline{T}}-\underline{\underline{T}}\,\underline{\underline{L}}\quad\text{上随流}$$

利用 $\dot{\underline{\underline{Q}}}$ 和 $\dot{\underline{\underline{Q}}}^T$ 的不同表达式组合，还可定义不同的导数，

$$\overset{\triangle}{\underline{\underline{T}}}=\dot{\underline{\underline{T}}}+\underline{\underline{L}}^T\underline{\underline{T}}+\underline{\underline{T}}\,\underline{\underline{L}}\quad\text{下随流}$$

$$\overset{\triangle}{\underline{\underline{T}}} = \dot{\underline{\underline{T}}} + \underline{\underline{L}}^T \underline{\underline{T}} + \underline{\underline{T}}\,\underline{\underline{L}} \quad \text{下随流}$$

△ 注意：上面提到的向量和张量场均为空间描述。考虑对空间描述量的物质导数

$$\frac{\partial}{\partial t} f_s(K(\bar{X},t),t) = \frac{\partial}{\partial t} f_s(\vec{r},t)\Big|_{\vec{r}=K(\bar{X},t)} + \frac{\partial}{\partial \vec{r}} \bar{f}(\vec{r},t)\Big|_{\vec{r}=K(\bar{X},t)} \vec{V}_m(\bar{X},t)$$

由之前关于速度、时间偏导数的结论可知，物质导数不保持客观性。故构建物质导数的共旋和随流导数。对向量场 $\vec{a}_s(K(\bar{X},t),t)$

$$\overset{\triangle}{\vec{a}}_s(K(\bar{X},t),t) = \left(\dot{\vec{a}}_s + \left(\frac{\partial}{\partial \vec{r}}\vec{a}_s\right)\vec{V}_m + \underline{\underline{L}}^T \vec{a}_s\right)\Big|_{\vec{r}=K(\bar{X},t)}, \quad \left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} + \vec{\nabla}\vec{v}\right)$$

$$\overset{\nabla}{\vec{a}}_s(K(\bar{X},t),t) = \left(\dot{\vec{a}}_s + \left(\frac{\partial}{\partial \vec{r}}\vec{a}_s\right)\vec{V}_m - \underline{\underline{L}}\,\vec{a}_s\right)\Big|_{\vec{r}=K(\bar{X},t)}, \quad \left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} - (\vec{\nabla}\vec{v})^T\right]$$

对张量场 $\underline{\underline{T}}_s(K(\bar{X},t),t)$

$$\overset{\triangle}{\underline{\underline{T}}}_s(K(\bar{X},t),t) = \left(\dot{\underline{\underline{T}}}_s + \left(\frac{\partial}{\partial \vec{r}}\underline{\underline{T}}_s\right)\vec{V}_m + \underline{\underline{L}}^T\underline{\underline{T}}_s + \underline{\underline{T}}_s\underline{\underline{L}}\right)\Big|_{\vec{r}=K(\bar{X},t)} \quad \text{下随体导数}$$

$$\overset{\nabla}{\underline{\underline{T}}}_s(K(\bar{X},t),t) = \left(\dot{\underline{\underline{T}}}_s + \left(\frac{\partial}{\partial \vec{r}}\underline{\underline{T}}_s\right)\vec{V}_m - \underline{\underline{L}}\,\underline{\underline{T}} - \underline{\underline{T}}\,\underline{\underline{L}}\right)\Big|_{\vec{r}=K(\bar{X},t)} \quad \text{上随体导数、}$$