标架不变性：物体的形变与流动

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△ 标架变换下的形变梯度张量：设(ϕ, ϕ^*)是一个标架变换，物体$\mathcal{B}$在两个标架下的参考构型为Ω_0, Ω_0^*，当前构型为Ω_t, Ω_t^*。物质点$X \in \mathcal{B}$在参考时刻下满足

$$\begin{cases} t_0^* = t_0 + a \\ \vec{X}^* = \vec{r}_0^*(t_0^*) + \underline{\underline{Q}}(t_0)(\vec{X} - \vec{r}_0) \end{cases}$$

在当前时刻下，

$$\begin{cases} t^* = t + a \\ \vec{x}^*(t^*) = \vec{r}_0^*(t^*) + \underline{\underline{Q}}(t)(\vec{x}(t) - \vec{r}_0) \end{cases}$$

设 $\chi = \kappa_t \circ \kappa_0^{-1}$，$\chi^* = \kappa_t^* \circ \kappa_0^{*-1}$ 分别是ϕ, ϕ^*下的形变映射。则当前时刻下

$$\chi^*(\vec{X}^*, t^*) = \vec{r}_0^*(t^*) + \underline{\underline{Q}}(t)(\chi(\vec{X}, t) - \vec{r}_0)$$

$$\underline{\underline{F}}^*(\vec{X}^*, t^*) = \frac{\partial}{\partial \vec{X}^*}\chi^*(\vec{X}^*, t^*) = \underline{\underline{Q}}(t)\frac{\partial}{\partial \vec{X}^*}\chi(\vec{X}, t)$$

$$= \underline{\underline{Q}}(t)\frac{\partial}{\partial \vec{X}}\chi(\vec{X}, t)\Big|_{\vec{X} = \vec{X}(\vec{X}^*)}\frac{\partial \vec{X}}{\partial \vec{X}^*}$$

由参考时刻的关系：

$$\vec{X} = \vec{r}_0 + \underline{\underline{Q}}^{-1}(t_0)\left(\vec{X}^* - \vec{r}_0^*(t_0^*)\right), \quad \frac{\partial \vec{X}}{\partial \vec{X}^*} = \underline{\underline{Q}}^{-1}(t_0) = \underline{\underline{Q}}^T(t_0)$$

$$\therefore \underline{\underline{F}}^*(\vec{X}^*, t) = \underline{\underline{Q}}(t)\,\underline{\underline{F}}(\vec{X}, t)\,\underline{\underline{Q}}^T(t_0) \neq \underline{\underline{Q}}(t)\,\underline{\underline{F}}(\vec{X}, t)\,\underline{\underline{Q}}^T(t)$$

故一般地，形变梯度张量不具有标架不变性。在刚体变换下由于有$\underline{\underline{Q}}(t) = \underline{\underline{Q}}(t_0) = \text{const}$，形变梯度张量具有标架变换不变性。

由于$\underline{\underline{Q}}(t_0)$本身是固定的，故常不失一般性地设$\underline{\underline{Q}}(t_0) = \underline{\underline{I}}$。即 $\underline{\underline{F}}^* = \underline{\underline{Q}}\,\underline{\underline{F}}$

△ 其他运动学张量：

$\underline{\underline{U}}^* = \underline{\underline{U}}$，$\underline{\underline{R}}^* = \underline{\underline{Q}}\,\underline{\underline{R}}$，$\underline{\underline{V}}^* = \underline{\underline{Q}}\,\underline{\underline{V}}\,\underline{\underline{Q}}^T$

$\underline{\underline{B}}^* = \underline{\underline{Q}}\,\underline{\underline{B}}\,\underline{\underline{Q}}^T$，$\underline{\underline{C}}^* = \underline{\underline{C}}$，

$\underline{\underline{L}}^* = \underline{\underline{Q}}\,\underline{\underline{L}}\,\underline{\underline{Q}}^T + \underline{\underline{A}}$　$\underline{\underline{D}}^* = \underline{\underline{Q}}\,\underline{\underline{D}}\,\underline{\underline{Q}}^T$，$\underline{\underline{W}}^* = \underline{\underline{Q}}\,\underline{\underline{W}}\,\underline{\underline{Q}}^T + \underline{\underline{A}}$

$\underline{\underline{U}}$，$\underline{\underline{C}}$是参考构型的函数，总客观，但表现为$\underline{\underline{U}}^* = \underline{\underline{U}}$，$\underline{\underline{C}}^* = \underline{\underline{C}}$，也是令$\underline{\underline{Q}}(t_0) = \underline{\underline{I}}$后的结果。一般地$\underline{\underline{U}}^* = \underline{\underline{Q}}(t_0)\,\underline{\underline{U}}\,\underline{\underline{Q}}^T(t_0)$，$\underline{\underline{C}}^* = \underline{\underline{Q}}(t_0)\,\underline{\underline{C}}\,\underline{\underline{Q}}^T(t_0)$

△ 假设物体$\mathcal{B}$的某部分在标架ϕ下，参考构型$\kappa_0(\mathcal{B})$内是一个曲面$\mathcal{C}$，$\vec{G}: \mathbb{R}^2 \supset \mathcal{U} \to \mathbb{R}^3$，$\vec{G}(\vec{u})$是曲面$\mathcal{C}$的参数方程。则面元$d\vec{A} = \frac{\partial \vec{G}}{\partial u_1} \times \frac{\partial \vec{G}}{\partial u_2}$。设由参考构型到当前构型的形变梯度张量为$\underline{\underline{F}}$，则当前构型面元$d\vec{a} = \underline{\underline{F}}\,\partial\vec{G}/\partial u_1 \times \underline{\underline{F}}\,\partial\vec{G}/\partial u_2$。在标架$\phi^*$下

$$d\vec{a}^* = \underline{\underline{F}}^*\,\partial\vec{G}/\partial u_1 \times \underline{\underline{F}}^*\,\partial\vec{G}/\partial u_2$$

$$= \underline{\underline{Q}}\,\underline{\underline{F}}\,\partial\vec{G}/\partial u_1 \times \underline{\underline{Q}}\,\underline{\underline{F}}\,\partial\vec{G}/\partial u_2 \quad （利用\ \underline{\underline{Q}}\vec{a} \times \underline{\underline{Q}}\vec{b} = \det\underline{\underline{Q}}\,\underline{\underline{Q}}^{-T}\,\vec{a}\times\vec{b}，且参考构型量是常量总客观）$$

$$= \det\underline{\underline{Q}}\,\underline{\underline{Q}}^T\,\underline{\underline{F}}\,\partial\vec{G}/\partial u_1 \times \underline{\underline{F}}\,\partial\vec{G}/\partial u_2$$

$$= \underline{\underline{Q}}\,d\vec{a}$$

故面元形变是客观的。具体地，物体构型曲面的法向量是客观的，即 $\vec{n}^* = \underline{\underline{Q}}\,\vec{n}$。