曲线坐标系4：球坐标与柱坐标下的微分

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Del in cylindrical and spherical coordinates

This is a list of some vector calculus formulae for working with common curvilinear coordinate systems.

Contents

Notes

- This article uses the standard notation ISO 80000-2, which supersedes ISO 31-11, for spherical coordinates (other sources may reverse the definitions of θ and φ):
  - The polar angle is denoted by $\theta \in [0, \pi]$: it is the angle between the z-axis and the radial vector connecting the origin to the point in question.
  - The azimuthal angle is denoted by $\varphi \in [0, 2\pi]$: it is the angle between the x-axis and the projection of the radial vector onto the xy-plane.
- The function atan2(y, x) can be used instead of the mathematical function arctan(y/x) owing to its domain and image. The classical arctan function has an image of (−π/2, +π/2), whereas atan2 is defined to have an image of (−π, π].

Coordinate conversions

Conversion between Cartesian, cylindrical, and spherical coordinates[1]

		From		
		Cartesian	Cylindrical	Spherical
To	Cartesian	N/A	$x = \rho\cos\varphi$ $y = \rho\sin\varphi$ $z = z$	$x = r\sin\theta\cos\varphi$ $y = r\sin\theta\sin\varphi$ $z = r\cos\theta$
	Cylindrical	$\rho = \sqrt{x^2+y^2}$ $\varphi = \arctan\left(\frac{y}{x}\right)$ $z = z$	N/A	$\rho = r\sin\theta$ $\varphi = \varphi$ $z = r\cos\theta$
	Spherical	$r = \sqrt{x^2+y^2+z^2}$ $\theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$ $\varphi = \arctan\left(\frac{y}{x}\right)$	$r = \sqrt{\rho^2+z^2}$ $\theta = \arctan\left(\frac{\rho}{z}\right)$ $\varphi = \varphi$	N/A

Unit vector conversions

Conversion between unit vectors in Cartesian, cylindrical, and spherical coordinate systems in terms of *destination* coordinates[1]

	Cartesian	Cylindrical	Spherical
Cartesian	N/A	$\hat{\mathbf{x}} = \cos\varphi\hat{\boldsymbol{\rho}} - \sin\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{y}} = \sin\varphi\hat{\boldsymbol{\rho}} + \cos\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{x}} = \sin\theta\cos\varphi\hat{\mathbf{r}} + \cos\theta\cos\varphi\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{y}} = \sin\theta\sin\varphi\hat{\mathbf{r}} + \cos\theta\sin\varphi\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$
Cylindrical	$\hat{\boldsymbol{\rho}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{\sqrt{x^2+y^2}}$ $\hat{\boldsymbol{\varphi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2+y^2}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	N/A	$\hat{\boldsymbol{\rho}} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$
Spherical	$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}}$ $\hat{\boldsymbol{\theta}} = \frac{\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}\right)z - \left(x^2+y^2\right)\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}$ $\hat{\boldsymbol{\varphi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2+y^2}}$	$\hat{\mathbf{r}} = \frac{\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}}{\sqrt{\rho^2+z^2}}$ $\hat{\boldsymbol{\theta}} = \frac{z\hat{\boldsymbol{\rho}} - \rho\hat{\mathbf{z}}}{\sqrt{\rho^2+z^2}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	N/A

Conversion between unit vectors in Cartesian, cylindrical, and spherical coordinate systems in terms of *source* coordinates

	Cartesian	Cylindrical	Spherical
Cartesian	N/A	$\hat{\mathbf{x}} = \frac{x\hat{\boldsymbol{\rho}} - y\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}}$ $\hat{\mathbf{y}} = \frac{y\hat{\boldsymbol{\rho}} + x\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{x}} = \frac{x\left(\sqrt{x^2+y^2}\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}\right) - y\sqrt{x^2+y^2+z^2}\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$ $\hat{\mathbf{y}} = \frac{y\left(\sqrt{x^2+y^2}\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}\right) + x\sqrt{x^2+y^2+z^2}\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$ $\hat{\mathbf{z}} = \frac{z\hat{\mathbf{r}} - \sqrt{x^2+y^2}\hat{\boldsymbol{\theta}}}{\sqrt{x^2+y^2+z^2}}$
Cylindrical	$\hat{\boldsymbol{\rho}} = \cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}$ $\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	N/A	$\hat{\boldsymbol{\rho}} = \frac{\rho\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}}{\sqrt{\rho^2+z^2}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \frac{z\hat{\mathbf{r}} - \rho\hat{\boldsymbol{\theta}}}{\sqrt{\rho^2+z^2}}$
Spherical	$\hat{\mathbf{r}} = \sin\theta\left(\cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}\right) + \cos\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\theta}} = \cos\theta\left(\cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}\right) - \sin\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}$	$\hat{\mathbf{r}} = \sin\theta\hat{\boldsymbol{\rho}} + \cos\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\theta}} = \cos\theta\hat{\boldsymbol{\rho}} - \sin\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	N/A

Del formula

Table with the del operator in cartesian, cylindrical and spherical coordinates

Operation	Cartesian coordinates (x, y, z)	Cylindrical coordinates (ρ, φ, z)	Spherical coordinates (r, θ, φ), where θ is the polar angle and φ is the azimuthal angle[α]
Vector field $\mathbf{A}$	$A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$	$A_\rho\hat{\boldsymbol{\rho}} + A_\varphi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$	$A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\varphi\hat{\boldsymbol{\varphi}}$
Gradient ∇f[1]	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$
Divergence $\nabla\cdot\mathbf{A}$[1]	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times\mathbf{A}$[1]	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}}$ $+\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}}$ $+\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$	$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}}$ $+\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}}$ $+\frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathbf{z}}$	$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\varphi\sin\theta) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}}$ $+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}}$ $+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$
Laplace operator $\nabla^2 f \equiv \Delta f$[1]	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$
Vector Laplacian $\nabla^2\mathbf{A} \equiv \Delta\mathbf{A}$[2]	$\nabla^2 A_x\hat{\mathbf{x}} + \nabla^2 A_y\hat{\mathbf{y}} + \nabla^2 A_z\hat{\mathbf{z}}$	$\left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\varphi}{\partial\varphi}\right)\hat{\boldsymbol{\rho}}$ $+\left(\nabla^2 A_\varphi - \frac{A_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\varphi}\right)\hat{\boldsymbol{\varphi}}$ $+\nabla^2 A_z\hat{\mathbf{z}}$	$\left(\nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}\right)\hat{\mathbf{r}}$ $+\left(\nabla^2 A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\varphi}{\partial\varphi}\right)\hat{\boldsymbol{\theta}}$ $+\left(\nabla^2 A_\varphi - \frac{A_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\varphi}\right)\hat{\boldsymbol{\varphi}}$
Material derivative[α][3] $(\mathbf{A}\cdot\nabla)\mathbf{B}$	$\mathbf{A}\cdot\nabla B_x\hat{\mathbf{x}} + \mathbf{A}\cdot\nabla B_y\hat{\mathbf{y}} + \mathbf{A}\cdot\nabla B_z\hat{\mathbf{z}}$	$\left(A_\rho\frac{\partial B_\rho}{\partial\rho} + \frac{A_\varphi}{\rho}\frac{\partial B_\rho}{\partial\varphi} + A_z\frac{\partial B_\rho}{\partial z} - \frac{A_\varphi B_\varphi}{\rho}\right)\hat{\boldsymbol{\rho}}$ $+\left(A_\rho\frac{\partial B_\varphi}{\partial\rho} + \frac{A_\varphi}{\rho}\frac{\partial B_\varphi}{\partial\varphi} + A_z\frac{\partial B_\varphi}{\partial z} + \frac{A_\varphi B_\rho}{\rho}\right)\hat{\boldsymbol{\varphi}}$ $+\left(A_\rho\frac{\partial B_z}{\partial\rho} + \frac{A_\varphi}{\rho}\frac{\partial B_z}{\partial\varphi} + A_z\frac{\partial B_z}{\partial z}\right)\hat{\mathbf{z}}$	$\left(A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial\theta} + \frac{A_\varphi}{r\sin\theta}\frac{\partial B_r}{\partial\varphi} - \frac{A_\theta B_\theta + A_\varphi B_\varphi}{r}\right)\hat{\mathbf{r}}$ $+\left(A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial\theta} + \frac{A_\varphi}{r\sin\theta}\frac{\partial B_\theta}{\partial\varphi} + \frac{A_\theta B_r}{r} - \frac{A_\varphi B_\varphi\cot\theta}{r}\right)\hat{\boldsymbol{\theta}}$ $+\left(A_r\frac{\partial B_\varphi}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\varphi}{\partial\theta} + \frac{A_\varphi}{r\sin\theta}\frac{\partial B_\varphi}{\partial\varphi} + \frac{A_\varphi B_r}{r} + \frac{A_\varphi B_\theta\cot\theta}{r}\right)\hat{\boldsymbol{\varphi}}$
Tensor $\nabla\cdot\mathbf{T}$ **(not confuse with 2nd order tensor divergence)**			

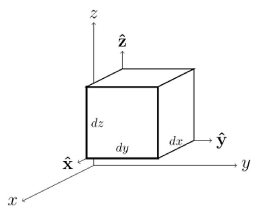
	Cartesian	Cylindrical	Spherical
	$\left(\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{yx}}{\partial y}+\frac{\partial T_{zx}}{\partial z}\right)\hat{\mathbf{x}}$ $+\left(\frac{\partial T_{xy}}{\partial x}+\frac{\partial T_{yy}}{\partial y}+\frac{\partial T_{zy}}{\partial z}\right)\hat{\mathbf{y}}$ $+\left(\frac{\partial T_{xz}}{\partial x}+\frac{\partial T_{yz}}{\partial y}+\frac{\partial T_{zz}}{\partial z}\right)\hat{\mathbf{z}}$	$\left[\frac{\partial T_{\rho\rho}}{\partial\rho}+\frac{1}{\rho}\frac{\partial T_{\varphi\rho}}{\partial\varphi}+\frac{\partial T_{z\rho}}{\partial z}+\frac{1}{\rho}(T_{\rho\rho}-T_{\varphi\varphi})\right]\hat{\boldsymbol{\rho}}$ $+\left[\frac{\partial T_{\rho\varphi}}{\partial\rho}+\frac{1}{\rho}\frac{\partial T_{\varphi\varphi}}{\partial\varphi}+\frac{\partial T_{z\varphi}}{\partial z}+\frac{1}{\rho}(T_{\rho\varphi}+T_{\varphi\rho})\right]\hat{\boldsymbol{\varphi}}$ $+\left[\frac{\partial T_{\rho z}}{\partial\rho}+\frac{1}{\rho}\frac{\partial T_{\varphi z}}{\partial\varphi}+\frac{\partial T_{zz}}{\partial z}+\frac{T_{\rho z}}{\rho}\right]\hat{\mathbf{z}}$	$\left[\frac{\partial T_{rr}}{\partial r}+2\frac{T_{rr}}{r}+\frac{1}{r}\frac{\partial T_{\theta r}}{\partial\theta}+\frac{\cot\theta}{r}T_{\theta r}+\frac{1}{r\sin\theta}\frac{\partial T_{\varphi r}}{\partial\varphi}-\frac{1}{r}(T_{\theta\theta}+T_{\varphi\varphi})\right]\hat{\mathbf{r}}$ $+\left[\frac{\partial T_{r\theta}}{\partial r}+2\frac{T_{r\theta}}{r}+\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta}+\frac{\cot\theta}{r}T_{\theta\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\varphi\theta}}{\partial\varphi}+\frac{T_{\theta r}}{r}-\frac{\cot\theta}{r}T_{\varphi\varphi}\right]\hat{\boldsymbol{\theta}}$ $+\left[\frac{\partial T_{r\varphi}}{\partial r}+2\frac{T_{r\varphi}}{r}+\frac{1}{r}\frac{\partial T_{\theta\varphi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\varphi\varphi}}{\partial\varphi}+\frac{T_{\varphi r}}{r}+\frac{\cot\theta}{r}(T_{\theta\varphi}+T_{\varphi\theta})\right]\hat{\boldsymbol{\varphi}}$
Differential displacement $d\boldsymbol{\ell}$[1]	$dx\,\hat{\mathbf{x}}+dy\,\hat{\mathbf{y}}+dz\,\hat{\mathbf{z}}$	$d\rho\,\hat{\boldsymbol{\rho}}+\rho\,d\varphi\,\hat{\boldsymbol{\varphi}}+dz\,\hat{\mathbf{z}}$	$dr\,\hat{\mathbf{r}}+r\,d\theta\,\hat{\boldsymbol{\theta}}+r\,\sin\theta\,d\varphi\,\hat{\boldsymbol{\varphi}}$
Differential normal area $d\mathbf{S}$	$dy\,dz\,\hat{\mathbf{x}}$ $+\,dx\,dz\,\hat{\mathbf{y}}$ $+\,dx\,dy\,\hat{\mathbf{z}}$	$\rho\,d\varphi\,dz\,\hat{\boldsymbol{\rho}}$ $+\,d\rho\,dz\,\hat{\boldsymbol{\varphi}}$ $+\,\rho\,d\rho\,d\varphi\,\hat{\mathbf{z}}$	$r^2\sin\theta\,d\theta\,d\varphi\,\hat{\mathbf{r}}$ $+\,r\sin\theta\,dr\,d\varphi\,\hat{\boldsymbol{\theta}}$ $+\,r\,dr\,d\theta\,\hat{\boldsymbol{\varphi}}$
Differential volume dV[1]	$dx\,dy\,dz$	$\rho\,d\rho\,d\varphi\,dz$	$r^2\sin\theta\,dr\,d\theta\,d\varphi$

^α This page uses θ for the polar angle and φ for the azimuthal angle, which is common notation in physics. The source that is used for these formulae uses θ for the azimuthal angle and φ for the polar angle, which is common mathematical notation. In order to get the mathematics formulae, switch θ and φ in the formulae shown in the table above.

Non-trivial calculation rules

1. $\operatorname{div}\,\operatorname{grad} f \equiv \nabla\cdot\nabla f \equiv \nabla^2 f$
2. $\operatorname{curl}\,\operatorname{grad} f \equiv \nabla\times\nabla f = \mathbf{0}$
3. $\operatorname{div}\,\operatorname{curl}\mathbf{A} \equiv \nabla\cdot(\nabla\times\mathbf{A}) = 0$
4. $\operatorname{curl}\,\operatorname{curl}\mathbf{A} \equiv \nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$ (Lagrange's formula for del)
5. $\nabla^2(fg) = f\nabla^2 g + 2\nabla f\cdot\nabla g + g\nabla^2 f$

Cartesian derivation



$$\operatorname{div} \mathbf{A} = \lim_{V \to 0} \frac{\iint_{\partial V} \mathbf{A} \cdot d\mathbf{S}}{\iiint_V dV} = \frac{A_x(x+dx)\,dy\,dz - A_x(x)\,dy\,dz + A_y(y+dy)\,dx\,dz - A_y(y)\,dx\,dz + A_z(z+dz)\,dx\,dy - A_z(z)\,dx\,dy}{dx\,dy\,dz}$$

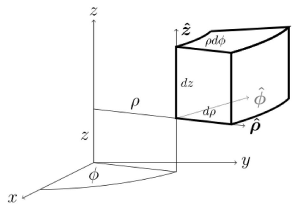
$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$(\operatorname{curl} \mathbf{A})_x = \lim_{S^{\perp \hat{\mathbf{x}}} \to 0} \frac{\int_{\partial S} \mathbf{A} \cdot d\ell}{\iint_S dS} = \frac{A_z(y+dy)\,dz - A_z(y)\,dz + A_y(z)\,dy - A_y(z+dz)\,dy}{dy\,dz}$$

$$= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

The expressions for $(\operatorname{curl} \mathbf{A})_y$ and $(\operatorname{curl} \mathbf{A})_z$ are found in the same way.

Cylindrical derivation



$$\operatorname{div} \mathbf{A} = \lim_{V \to 0} \frac{\iint_{\partial V} \mathbf{A} \cdot d\mathbf{S}}{\iiint_V dV}$$

$$= \frac{A_\rho(\rho+d\rho)(\rho+d\rho)\,d\phi\,dz - A_\rho(\rho)\rho\,d\phi\,dz + A_\phi(\phi+d\phi)\,d\rho\,dz - A_\phi(\phi)\,d\rho\,dz + A_z(z+dz)\,d\rho\,(\rho+d\rho/2)\,d\phi - A_z(z)\,d\rho(\rho+d\rho/2)\,d\phi}{\rho\,d\phi\,d\rho\,dz}$$

$$= \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

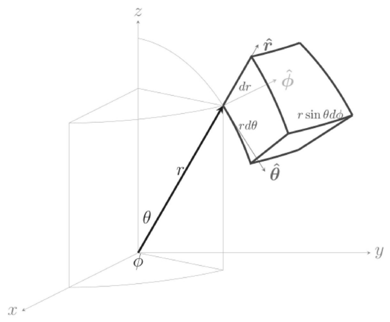
$$
\begin{aligned}
(\operatorname{curl}\mathbf{A})_\rho &= \lim_{S^{\perp\hat{\boldsymbol\rho}}\to 0} \frac{\int_{\partial S}\mathbf{A}\cdot d\ell}{\iint_S dS} \\
&= \frac{A_\phi(z)(\rho+d\rho)\,d\phi - A_\phi(z+dz)(\rho+d\rho)\,d\phi + A_z(\phi+d\phi)\,dz - A_z(\phi)\,dz}{(\rho+d\rho)\,d\phi\,dz} \\
&= -\frac{\partial A_\phi}{\partial z} + \frac{1}{\rho}\frac{\partial A_z}{\partial \phi}
\end{aligned}
$$

$$
\begin{aligned}
(\operatorname{curl}\mathbf{A})_\phi &= \lim_{S^{\perp\hat{\boldsymbol\phi}}\to 0} \frac{\int_{\partial S}\mathbf{A}\cdot d\ell}{\iint_S dS} \\
&= \frac{A_z(\rho)\,dz - A_z(\rho+d\rho)\,dz + A_\rho(z+dz)\,d\rho - A_\rho(z)\,d\rho}{d\rho\,dz} \\
&= -\frac{\partial A_z}{\partial \rho} + \frac{\partial A_\rho}{\partial z}
\end{aligned}
$$

$$
\begin{aligned}
(\operatorname{curl}\mathbf{A})_z &= \lim_{S^{\perp\hat{\mathbf z}}\to 0} \frac{\int_{\partial S}\mathbf{A}\cdot d\ell}{\iint_S dS} \\
&= \frac{A_\rho(\phi)\,d\rho - A_\rho(\phi+d\phi)\,d\rho + A_\phi(\rho+d\rho)(\rho+d\rho)\,d\phi - A_\phi(\rho)\rho\,d\phi}{\rho\,d\rho\,d\phi} \\
&= -\frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi} + \frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial \rho}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{curl}\mathbf{A} &= (\operatorname{curl}\mathbf{A})_\rho\hat{\boldsymbol\rho} + (\operatorname{curl}\mathbf{A})_\phi\hat{\boldsymbol\phi} + (\operatorname{curl}\mathbf{A})_z\hat{\mathbf z} \\
&= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol\phi} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)\hat{\mathbf z}
\end{aligned}
$$

Spherical derivation



$$\begin{aligned}
\operatorname{div}\mathbf{A} &= \lim_{V\to 0}\frac{\iint_{\partial V}\mathbf{A}\cdot d\mathbf{S}}{\iiint_V dV} \\
&= \frac{A_r(r+dr)(r+dr)\,d\theta\,(r+dr)\sin\theta\,d\phi - A_r(r)r\,d\theta\,r\sin\theta\,d\phi + A_\theta(\theta+d\theta)\sin(\theta+d\theta)r\,dr\,d\phi - A_\theta(\theta)\sin(\theta)r\,dr\,d\phi + A_\phi(\phi+d\phi)r\,dr\,d\theta - A_\phi(\phi)r\,dr\,d\theta}{dr\,r\,d\theta\,r\sin\theta\,d\phi} \\
&= \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}
\end{aligned}$$

$$\begin{aligned}
(\operatorname{curl}\mathbf{A})_r = \lim_{S^{\perp\hat{r}}\to 0}\frac{\int_{\partial S}\mathbf{A}\cdot d\ell}{\iint_S dS} &= \frac{A_\theta(\phi)r\,d\theta + A_\phi(\theta+d\theta)r\sin(\theta+d\theta)\,d\phi - A_\theta(\phi+d\phi)r\,d\theta - A_\phi(\theta)r\sin(\theta)\,d\phi}{r\,d\theta\,r\sin\theta\,d\phi} \\
&= \frac{1}{r\sin\theta}\frac{\partial(A_\phi\sin\theta)}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}
\end{aligned}$$

$$\begin{aligned}
(\operatorname{curl}\mathbf{A})_\theta = \lim_{S^{\perp\hat{\theta}}\to 0}\frac{\int_{\partial S}\mathbf{A}\cdot d\ell}{\iint_S dS} &= \frac{A_\phi(r)r\sin\theta\,d\phi + A_r(\phi+d\phi)\,dr - A_\phi(r+dr)(r+dr)\sin\theta\,d\phi - A_r(\phi)\,dr}{dr\,r\sin\theta\,d\phi} \\
&= \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}
\end{aligned}$$

$$\begin{aligned}
(\operatorname{curl}\mathbf{A})_\phi = \lim_{S^{\perp\hat{\phi}}\to 0}\frac{\int_{\partial S}\mathbf{A}\cdot d\ell}{\iint_S dS} &= \frac{A_r(\theta)\,dr + A_\theta(r+dr)(r+dr)\,d\theta - A_r(\theta+d\theta)\,dr - A_\theta(r)r\,d\theta}{r\,dr\,d\theta} \\
&= \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta}
\end{aligned}$$

$$\operatorname{curl}\mathbf{A} = (\operatorname{curl}\mathbf{A})_r\,\hat{\boldsymbol r} + (\operatorname{curl}\mathbf{A})_\theta\,\hat{\boldsymbol\theta} + (\operatorname{curl}\mathbf{A})_\phi\,\hat{\boldsymbol\phi} = \frac{1}{r\sin\theta}\left(\frac{\partial(A_\phi\sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\hat{\boldsymbol r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\hat{\boldsymbol\theta} + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol\phi}$$

Unit vector conversion formula

The unit vector of a coordinate parameter u is defined in such a way that a small positive change in u causes the position vector $\vec{\boldsymbol r}$ to change in $\hat{\boldsymbol u}$ direction.

Therefore,

$$\frac{\partial\vec{\boldsymbol r}}{\partial u} = \frac{\partial s}{\partial u}\hat{\boldsymbol u}$$

where s is the arc length parameter.

For two sets of coordinate systems u_i and v_j, according to chain rule,

$$d\vec{\boldsymbol r} = \sum_i \frac{\partial\vec{\boldsymbol r}}{\partial u_i}du_i = \sum_i \frac{\partial s}{\partial u_i}\hat{\boldsymbol u}_i du_i = \sum_j \frac{\partial s}{\partial v_j}\hat{\boldsymbol v}_j dv_j = \sum_j \frac{\partial s}{\partial v_j}\hat{\boldsymbol v}_j \sum_i \frac{\partial v_j}{\partial u_i}du_i = \sum_i\sum_j \frac{\partial s}{\partial v_j}\frac{\partial v_j}{\partial u_i}\hat{\boldsymbol v}_j du_i$$

Now, we isolate the i^{th} component. For $i \neq k$, let $du_k = 0$. Then divide on both sides by du_i to get:

$$\frac{\partial s}{\partial u_i}\hat{\boldsymbol u}_i = \sum_j \frac{\partial s}{\partial v_j}\frac{\partial v_j}{\partial u_i}\hat{\boldsymbol v}_j$$

See also

- Del
- Orthogonal coordinates
- Curvilinear coordinates
- Vector fields in cylindrical and spherical coordinates

References

1. Griffiths, David J. (2012). *Introduction to Electrodynamics*. Pearson. ISBN 978-0-321-85656-2.
2. Arfken, George; Weber, Hans; Harris, Frank (2012). *Mathematical Methods for Physicists* (Seventh ed.). Academic Press. p. 192. ISBN 9789381269558.
3. Weisstein, Eric W. "Convective Operator" (http://mathworld.wolfram.com/ConvectiveOperator.html). *Mathworld*. Retrieved 23 March 2011.

External links

- Maxima Computer Algebra system scripts (http://www.csulb.edu/~woollett/) to generate some of these operators in cylindrical and spherical coordinates.