应变率

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△设物体B进行由映射χ描述的运动. 对空间描述的速度$\vec{v}_s(\vec{r},t)$求关于空间位置$\vec{r}$的导数:

$$\underline{\underline{L}} \overset{\text{def}}{=} \frac{\partial \vec{v}_s(\vec{r},t)}{\partial \vec{r}} = \underline{\underline{L}}(\vec{r},t)$$

得到的线性算符$\underline{\underline{L}}$称物体B在此运动的速度梯度张量(velocity gradient tensor).
记

$$\underline{\underline{L}} \equiv \tfrac{1}{2}(\underline{\underline{L}}+\underline{\underline{L}}^*)+\tfrac{1}{2}(\underline{\underline{L}}-\underline{\underline{L}}^*) = \underline{\underline{D}}+\underline{\underline{W}}, \quad \underline{\underline{D}} \overset{\text{def}}{=} \tfrac{1}{2}(\underline{\underline{L}}+\underline{\underline{L}}^*), \quad \underline{\underline{W}} \overset{\text{def}}{=} \tfrac{1}{2}(\underline{\underline{L}}-\underline{\underline{L}}^*)$$

则自伴随算符$\underline{\underline{D}}$称为应变率张量(rate of strain tensor), 反自伴随算符$\underline{\underline{W}}$称为旋转张量(spin tensor)

△速度梯度张量、应变率张量和旋转张量在上述定义下都是空间描述量.

定理: $\underline{\underline{L}} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$, $\dot{\underline{\underline{C}}} = \underline{\underline{F}}^*\underline{\underline{F}}^*(2\underline{\underline{D}})$, $\dot{\underline{\underline{B}}} = \underline{\underline{L}}\,\underline{\underline{B}}+\underline{\underline{B}}\,\underline{\underline{L}}^*$

证明: 由
$$\frac{\partial}{\partial \vec{X}}\vec{v}_m(\vec{X},t) = \frac{\partial}{\partial \vec{X}}\left[\frac{\partial}{\partial t}\chi(\vec{X},t)\right] = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial \vec{X}}\chi(\vec{X},t)\right] = \frac{\partial}{\partial t}\underline{\underline{F}}(\vec{X},t) \equiv \dot{\underline{\underline{F}}}(\vec{X},t)$$

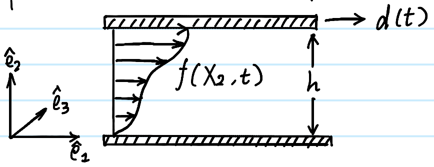
有
$$\underline{\underline{L}}(\vec{r},t) = \frac{\partial}{\partial \vec{r}}\vec{v}_s(\vec{r},t) = \frac{\partial}{\partial \vec{r}}\vec{v}_m(\chi_t^{-1}(\vec{r}),t) = \underbrace{\frac{\partial}{\partial \vec{X}}\vec{v}_m(\vec{X},t)\Big|_{\vec{X}=\chi_t^{-1}(\vec{r})}}_{=\dot{\underline{\underline{F}}}\ \text{(代入上式)}}\ \underbrace{\frac{\partial \chi_t^{-1}(\vec{r})}{\partial \vec{r}}}_{=\underline{\underline{F}}^{-1}\ \text{(反函数定理)}} \quad \text{(链式法则)}$$
$$= \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$$

$$\begin{aligned}\dot{\underline{\underline{C}}} &= \frac{\partial}{\partial t}(\underline{\underline{F}}^*\underline{\underline{F}}) = (\dot{\underline{\underline{F}}}^*\underline{\underline{F}} + \underline{\underline{F}}^*\dot{\underline{\underline{F}}}) \\ &= (\underline{\underline{L}}\,\underline{\underline{F}})^*\underline{\underline{F}} + \underline{\underline{F}}^*(\underline{\underline{L}}\,\underline{\underline{F}}) \\ &= \underline{\underline{F}}^*\underline{\underline{F}}^*\underline{\underline{L}} + \underline{\underline{F}}^*\underline{\underline{F}}^*\underline{\underline{L}}^* \\ &= \underline{\underline{F}}\,\underline{\underline{F}}^*(2\underline{\underline{D}})\end{aligned}$$

$$\begin{aligned}\dot{\underline{\underline{B}}} &= \frac{\partial}{\partial t}(\underline{\underline{F}}\,\underline{\underline{F}}^*) = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^* + \underline{\underline{F}}\,\dot{\underline{\underline{F}}}^* \\ &= \underline{\underline{L}}\,\underline{\underline{F}}\,\underline{\underline{F}}^* + \underline{\underline{F}}\,\underline{\underline{F}}^*\underline{\underline{L}}^* \\ &= \underline{\underline{L}}\,\underline{\underline{B}} + \underline{\underline{B}}\,\underline{\underline{L}}^*\end{aligned}$$

□

例: 库埃特流(Couette flow):



$$\chi_t: \begin{cases} x_1 = X_1 + f(X_2,t) \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$ 假设了层流.

函数 $f(X_2,t) = \begin{cases} d(t), & X_2 = h \\ 0, & X_2 = 0 \end{cases}$
假设了无滑(no-slip)边界条件.

$$\underline{\underline{F}}(\vec{X},t) = \begin{pmatrix} 1 & f'_{X_2}(X_2,t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$f'_{X_2} \equiv \partial f/\partial X_2$

$f'_{X_2}(h,t) = \dot{d}(t)$, $f'_X(0,t) = 0$.

若 $f(X_2,t) = \frac{d(t)}{h}X_2$, 记 $\gamma(t) \overset{\text{def}}{=} d(t)/h$ 称表观剪切应度, 则 $x_1 = X_1 + \gamma(t)X_2$

$$\underline{\underline{F}} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$, 不依赖$\vec{X}$, 该形变是均匀形变.

$$\begin{pmatrix} \cdots \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{B}} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \underline{\underline{C}} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{L}} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1} = \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{D}} = \frac{1}{2}\left(\underline{\underline{L}} + \underline{\underline{L}}^{T}\right) = \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

其中 $\dot{\gamma}(t)$ 是表观应变速率